The Hopfian Property of n-Periodic Products of Groups

S. I. Adian V.S.Atabekyan

Abstract

Let H be a subgroup of a group G. A normal subgroup N_H of H is said to be *inheritably normal* if there is a normal subgroup N_G of G such that $N_H = N_G \cap H$. It is proved in the paper that a subgroup N_{G_i} of a factor G_i of the n-periodic product $\prod_{i\in I}^n G_i$ with nontrivial factors G_i is an inheritably normal subgroup if and only if N_{G_i} contains the subgroup G_i^n. It is also proved that for odd $n \geq 665$ every nontrivial normal subgroup in a given n-periodic product $G = \prod_{i\in I}^n G_i$ contains the subgroup G^n. It follows that almost all n-periodic products $G = G_1 \overset{n}{*} G_2$ are Hopfian, i.e., they are not isomorphic to any of their proper quotient groups. This allows one to construct nonsimple and not residually finite Hopfian groups of bounded exponents.

Introduction. The notion of periodic product of period n (or n-periodic product) for a given family of groups $\{G_i\}_{i\in I}$ (denoted by $\prod_{i\in I}^n G_i$), was introduced in 1976 by the first author of this paper in [1]. It led to a solution of the well-known problem by Maltsev on the existence of a product operation of groups different from the classical operations of direct or free products of groups and possessing of all of the natural properties of these operations, including the so-called hereditary property for subgroups. The last property was named *Maltsev's postulate* in connection with this Maltsev problem.

The periodic product of given period n (or n-periodic product) of a given family of groups $\{G_i\}_{i\in I}$ is defined for any odd $n \geq 665$ on the basis of the Novikov-Adian theory thoroughly explained in the monograph [3] (see also [2]). This group $\prod_{i\in I}^n G_i$

is defined in the class of all groups. It is the quotient group of the free product $\{G_i\}_{i\in I}$ of the given family of groups by a specially chosen system of defining relations of the form $A^n = 1$. These product operations possess the main properties of the classical direct and free product of groups. They are exact and associative and have the hereditary property for subgroups. The last property means that any subgroups H_i in the factors G_i of the n-periodic product $F = \prod_{i\in I}^n G_i$ of the family of groups $\{G_i\}_{i\in I}$ generate their own n-periodic product in the group $\prod_{i\in I}^n G_i$, the identical embeddings $H_i \to G_i$ can be extended to an embedding of the n-periodic product $\prod_{i\in I}^n H_i$ of the family of subgroups $\{H_i\}_{i\in I}$ in the n-periodic product $\prod_{i\in I}^n G_i$, i.e., the subgroups of the factors $\{G_i\}_{i\in I}$ generate in $\prod_{i\in I}^n G_i$ its own n-periodic product.

The construction of the n-periodic product of odd period n introduced in [1] by Adian has also the following important property of so-called *conditional periodicity*, which can be regarded as a natural analog of the commutation of elements from different factors in the direct products of groups: *If the original groups G_i do not contain involutions, then the new operation of groups $\prod_{i\in I}^n G_i$ can be constructed in such a way that, in them, the equality $x^n = 1$ holds for any x which is not conjugating to elements of the original factors* . This property allowed to prove in 1978 (see [5] and [2]) the following interesting simplicity criterion of n-periodic products of groups without involutions.

Theorem 1. *An n-periodic product of odd period $n \geq 665$ of a given family of groups without involutions $\{G_i\}_{i\in I}$ is a simple group if and only if for each factor G_i the equality $G_i^n = G_i$ holds.*

This criterion of simplicity allowed to construct in [5] a new series of finitely generated infinite simple groups in varieties of periodic groups of odd composite periods nk for $n \geq 665$ and $k > 1$. Thus, a positive answer to the following question von H. Neumann's monograph was given: *Is it possible for a variety, different from the variety of all groups, to contain infinitely many nonisomorphic non-Abelian simple groups*?

The present work can be regarded as a continuation of the papers [1], [5], and [2]. Here we investigate some properties of normal subgroups of n-periodic products of groups. For instance, we consider the interesting property of the *extendability of a congruence, presented on a given subgroup G_i, to some congruence on the group G.*

This means that the quotient group of a subgroup G_i of G is naturally embedded into quotient group of the group G.

Definition 1. *A normal subgroup N_H of a subgroup H of G is said to be inheritably normal if there exists a normal subgroup N_G of G such that $H \cap N_G = N_H$.*

If *any* normal subgroup of a given subgroup H of G is inheritably normal, then the subgroup H is called *inheritably factorizable.* The concept of an inheritably factorizable subgroup was introduced by B. Neumann in [6], where such subgroups were called *E-subgroups.*

In the paper [7], inheritably normal free subgroups of infinite rank in the free group of rank 2 were constructed. It was proved in [8] that every noncyclic subgroup of the free Burnside group $\mathbf{B}(m, n)$ of odd period $n \geq 1003$ contains an inheritably factorizable subgroup isomorphic to the free Burnside group of infinite rank $\mathbf{B}(\infty, n)$.

For the n-periodic product of two given factors G_1 and G_2, we use the notation $G_1 \overset{n}{*} G_2$.

In this paper, we prove a necessary and sufficient condition for a given normal subgroup of a factor G_i of the nontrivial n-periodic product $\prod_{i \in I}^{n} G_i$ of an arbitrary family of groups $\{G_i\}_{i \in I}$ to be an inheritably normal subgroup.

It is also proved that any nontrivial normal subgroup of an n-periodic product $G = \prod_{i \in I}^{n} G_i$ contains the subgroup G^n. It follows that if at least in one of the factors of the n-periodic product $G = \prod_{i \in I}^{n} G_i$ the identity $x^n = 1$ does not hold, then G is a Hopfian group, i.e., it is not isomorphic to any of its proper quotient groups. This allows us to construct nonsimple and not residually finite Hopfian groups of bounded period.

Description of inheritably normal subgroups in factors of periodic products. Recall that the n-periodic product $G_1 \overset{n}{*} G_2$ of two given nontrivial factors is obtained from the free product $G_1 * G_2$ by adding defining relations of the form $A^n = 1$ for all elementary periods $A \in (G_1 * G_2)$ for all ranks $\alpha \geqslant 1$. In particular, any cyclicly reduced in $G_1 * G_2$ word A of length $|A| > 1$, which is not the product of two involutions and does not contain a 9-power of shorter words, is an elementary period of rank 1.

We need the following simple lemma.

Lemma 1. *If in the free product $G_1 * G_2$ a given cyclically reduced word of a length ≥ 2 is the product of two involutions, then some cyclic shift of this product has the normal form $c_1 z c_2 z^{-1}$, where c_1 and c_2 are involutions in G_1 or G_2.*

Proof. It is well known that every element of finite order of the free product $G_1 * G_2$ is conjugate to an element of G_1 or of G_2. Let

$$c = (a_1...a_t)c_1(a_1...a_t)^{-1} \qquad \text{and} \qquad d = (h_1...h_s)c_2(h_1...h_s)^{-1}$$

be the normal forms of two involutions of $G_1 * G_2$, where c_1 and c_2 are some involutions from the factors G_1 or G_2. If the cyclically reduced form of the element cd is of length ≥ 2, then it is conjugate to either $c_1 c_2$ or an element of the form

$$c_1 z_1 z_2 ... z_k c_2 z_k^{-1} ... z_2^{-1} z_1^{-1},$$

where $z_1 z_2 ... z_k$ is the normal form of $(a_1...a_t)^{-1} h_1 ... h_s$.

The element $c_1 z_1 z_2 ... z_k c_2 z_k^{-1} ... z_2^{-1} z_1^{-1}$ can be assumed to be irreducible. For instance, if c_1 and z_1 belong to the same group G_j, then, using the notation $c_1' = z_1^{-1} c_1 z_1$, we can replace the element $c_1 z_1 z_2 ... z_k c_2 z_k^{-1} ... z_2^{-1} z_1^{-1}$ by $c_1' z_2 ... z_k c_2 z_k^{-1} ... z_2^{-1}$. Lemma 1 is proved. ☐

Theorem 2. *A nontrivial normal subgroup N_{G_i} of the factor G_i of a nontrivial n-periodic product $G = \prod_{i \in I}^{n} G_i$ is an inheritably normal subgroup of G_i if and only if N_{G_i} contains all the nth powers of elements of G_i, i.e., the following inclusion $G_i^n \subset N_{G_i}$ holds.*

Proof. Obviously, it is sufficient to consider only the case of an n-periodic product of two nontrivial groups $G_1 \overset{n}{*} G_2$ and to prove our statement for the factor G_1.

Suppose that N_{G_1} is a an inheritably normal subgroup of the factor G_1, that is N_{G_1} is a nontrivial normal subgroup of G_1 and N is a normal subgroup of G such that the equality $N_{G_1} = N \cap G_1$ holds. It is sufficient to prove $g^n \in N_{G_1}$ for an arbitrary nontrivial element $g \in G_1$.

If the order $|G_1|$ of the group G_1 equals either 2 or 3, then G_1 is a simple group and $N_{G_1} = G_1$.

Hence we can assume that $|G_1| > 3$.

Let g be an arbitrary nontrivial element of G_1.

We consider separately the following two cases:

1. $|G_2| \geq 3$ and 2. $|G_2| = 2$.

Case 1. $|G_2| \geq 3$.

We choose nontrivial elements $a \in N_{G_1}$ and $b_1, b_2 \in G_2$, where $b_1 \neq b_2$.

Consider the element $b_1^{-1}ab_1b_2^{-1}ab_2g$. In the free product $G_1 * G_2$, it has the cyclically irreducible normal form $b_1^{-1}a(b_1b_2^{-1})ab_2g$ of length 6 in the free product $G_1 * G_2$. Using Lemma 1, we can check that it is not equal to the product of two involutions in $G_1 * G_2$. By the definition of the n-periodic product, the word $b_1^{-1}a(b_1b_2^{-1})ab_2g$ is an elementary period of rank 1 and, therefore, in the group $G = \prod_{i\in I}^n G_i$, the defining relation $(b_1^{-1}a(b_1b_2^{-1})ab_2g)^n = 1$ holds.

It follows from the condition $a \in N_{G_1}$ and $N_{G_1} \subset N$ that

$$(b_1^{-1}ab_1b_2^{-1}ab_2g)^n \equiv g^n \pmod N.$$

As was mentioned above, the relation $(b_1^{-1}ab_1b_2^{-1}ab_2g)^n = 1$ holds in G. Consequently, we have $g^n \equiv 1 \pmod N$. This means that $g^n \in N$. Hence $g^n \in N_{G_1}$, because $g \in G_1$.

Thus, the condition $g \in G_1$ implies $g^n \in N_{G_1}$. Therefore, the necessity of the condition $G_1^n \subset N_{G_1}$ is proved in Case 1.

Case 2. $|G_2| = 2$. In this case, G_2 is generated by some involution b.

Assume $|N_{G_1}| \geq 3$. Then there exist nontrivial elements $a_1, a_2 \in N_{G_1}$ such that $a_1 \neq a_2$. Therefore, according Lemma 1, the element $ba_1ba_1ba_2bg$ is not equal to the product of two involutions in the free product $G_1 * G_2$. Hence the word $ba_1ba_1ba_2bg$ is an elementary period of rank 1 and the defining relation $(ba_1ba_1ba_2b)^n = 1$ holds in the group $G = G_1 *^n G_2$.

On the other hand, it follows from $a_1, a_2 \in N_{G_1}$ that $(ba_1b)a_1(ba_2b) \in N_{G_1}$. Since $N_{G_1} \subset N$, we obtain

$$(ba_1ba_1ba_2bg)^n \equiv g^n \pmod N.$$

Consequently, the relation $g^n \in N \cap G_1 = N_{G_1}$ holds.

It remains to consider the subcase $|N_{G_1}| = 2$ for Case 1. In this subcase, N_{G_1} is generated by some involution a.

Reasoning as above, we can prove that if $g^2 \neq 1$, then $babg$ is an elementary period of rank 1. Hence $(babg)^n \equiv g^n \pmod N$ and we obtain $g^n \in N_{G_1}$.

If $g^2 = 1$ in G, we consider the word $babagbgbg$, which is also an elementary period of rank 1 and hence a defining relation $(babagbgbg)^n = 1$ holds in G.

Using the conditions $(bab), a \in N$ and the equations $g^2 = 1 = b^2$ in G, we obtain the relation $(babagbgbg)^n \equiv gbg^n bg \pmod{N}$. Then, using the defining relation $(babagbgbg)^n = 1$, we conclude that $g^n \in N$.

Consequently, we obtain $g^n \in N \cap G_1 = N_{G_1}$.

The first part of Theorem 2 is proved.

To prove the second part of Theorem 2, we assume that a nontrivial normal subgroup N_{G_1} of the group G_1 contains G_1^n.

Let us show that N_{G_1} is an inheritably normal subgroup of G_1.

Let N_2 be the normal closure of the subgroup G_2 in the group $G = G_1 \overset{n}{*} G_2$. The quotient group $G_1 \overset{n}{*} G_2/N_2$ is obtained from the group $G_1 \overset{n}{*} G_2$ by adding new defining relations $g = 1$ for all $g \in G_2$. Therefore, every defining relation $A^n = 1$ can be replaced by a new relation of the form $A_1^n = 1$, where $A_1 \in G_1$ is obtained from A in $A^n = 1$ by deleting all letters of the group G_2. The quotient group $G_1 \overset{n}{*} G_2/N$ is isomorphic to the quotient group G_1/N_1 by a normal subgroup N_1 which is the normal closure of a set of words of the form A_1^n in the group G_1. This means that N_1 is contained in G_1^n. Therefore, we have $G_1 \cap N_2 = N_1 \subset G_1^n$.

According to our assumption, the relation $G_1^n \subset N_{G_1}$ holds. Therefore, we have the equality $G_1 \cap N_{G_1} N_2 = N_{G_1}(G_1 \cap N_2) = N_{G_1}$. Since N_{G_1} is a normal subgroup of G_1 and N_2 is the normal closure of the group G_2, the product $N = N_{G_1} N_2$ is a normal subgroup of $G_1 \overset{n}{*} G_2$. Thus, from the equality $G_1 \cap N_{G_1} N_2 = N_{G_1}$, it follows that N_{G_1} is an inheritably normal subgroup of G_1 in the n-periodic product $G_1 \overset{n}{*} G_2$. Theorem 2 is proved. $\square$

From the proof of Theorem 2, we also have the following.

Corrolary 1. *Let G_1 be an inheritably factorizable subgroup of the n-periodic product $G_1 \overset{n}{*} \mathbf{Z}_2$, where $n \geq 665$ is odd. If G_1 contains some involution, then it is the unique involution of G_1 that belongs to the center of G_1.*

The following statement about inheritably factorizable subgroups of n-periodic products of groups is a generalization of Theorem 1 from the paper [9] by the second author.

Corrolary 2. *In any n-periodic product $G = G_1 *^n G_2$ of nontrivial components G_1 and G_2 each factor G_i is an inheritably factorizable subgroup in G if and only if every nontrivial normal subgroup N_{G_i} of G_i contains the subgroup G_1^n.*

Normal subgroups of n-periodic products. A slight modification of the proof given in [1] of Theorem 1 on the simplicity criterion of n-periodic products of odd exponents $n \geq 665$ for groups without involutions allows us to obtain the following generalization of that theorem.

In what follows, we assume that n is a fixed odd number and $n \geqslant 665$.

Theorem 3. *If the factors $G_i, i \in I$ in the given n-periodic product $F = \prod_{i\in I}^n G_i$ for odd $n \geqslant 665$ do not contain involutions, then any nontrivial normal subgroup N of the group F contains the subgroup F^n.*

Proof. Consider a nontrivial normal subgroup N of F and let E be an arbitrary nontrivial element in N.

According to Theorem 7 of the paper [1], either the word E is conjugate in F to some element $a \in G_i$ for some $i \in I$ or E is conjugate to some word of the form A^r, where A is an elementary period of some rank γ, which depends on the word E.

Assume that the given word $E \in N$ is conjugate in F to some element $a \in G_k$ for some $k \in I$. Then $a \in N$. Let us to prove that $G_j^n \subset N$ for every $j \in I$. Clearly, we have $a \in N_{G_k} = N \cap G_k$. Since N_{G_k} is a nontrivial inheritably normal subgroup in G_k, it follows that, by Theorem 2, we have the inclusion $G_k^n \subset N$. By the definition of the n-periodic product, for any nontrivial element $b \in G_j$, where $j \neq k$, the element ab is an elementary period of rank 1, and hence we have the defining relation $(ab)^n = 1$ in F. From the relation $a \in N$ and the equality $(ab)^n = 1$, it follows that $b^n \in N$, i.e., the inclusion $G_j^n \subset N$ holds for all $j \neq k$ as well.

Thus, we have proved that if N contains some element a from some group G_k, then for all $j \in I$ the inclusion $G_j^n \subset N$ holds.

It remains to consider the case when the word $E \in N$ is not conjugate in F to any element $a \in G_i$ for all $i \in I$. In this case, by Lemma 4 from [1], the word $E \in N$ conjugates to some power A^r of some elementary period A of some rank γ, where $0 < \gamma \leqslant \alpha$ and $0 < r \leqslant (n+1)/2 + 46$. By the same lemma, one can assume

that the word A^q occurs into some word which belongs to the class $\mathcal{M}_{\gamma-1}$. Then, by Lemma [3, II.6.13], we obtain $A^q \in \mathcal{M}_{\gamma-1}$.

We will prove that, in the remaining case, the relation $G_i^n \subset N$ holds for any $i \in I$. It follows from $E \in N$ that $A^{kr} \in N$ for any k. Hence there exists a number t such that $A^t \in N$ and $(n/3 \leq t \leq 2n/3)$.

Consider the word aA^t, where a is an arbitrary nontrivial element of some group G_i, $(i \in I)$. Let $D = [a, A^t]_0$ be the normal form of the word aA^t.

From $A^q \in \mathcal{M}_{\gamma-1}$, by virtue of Lemma [3, II.6.13], we obtain $A^t \in \mathcal{M}_{\gamma-1}$. Therefore $D \in \mathcal{L}_{\gamma-1}$.

Since A is an elementary period of rank γ and the inequality $t \geqslant n/3$ holds, then the word D contains only one active kernel V of rank γ with period A, which by [3, IV.1.7] contains not less than $n/3 - 44$ and not more than $2n/3$ segments. At so, by [3, I.4.34], it follows from $D \in \mathcal{L}_{\gamma-1}$ that $D \in \mathcal{A}_{\gamma+1}$.

By Lemma 4 from [1], one of the following two cases holds:

1) $D = SyS^{-1}$ in F for some $S \in A_{\gamma+3}$, where y is an element of one of the groups G_i;

2) the word D is conjugate in F to some power C^l, where C is an elementary period of some rank.

We shall prove that Case 1) is impossible. Suppose that

$$D = SyS^{-1} \text{ in } F, \tag{1}$$

where $S \in \mathcal{A}_{\gamma+3}$ and $y \in G_i$ for some $i \in I$. Since $|y| = 1$, we assume that $SyS^{-1} \in \mathcal{R}_0$. By virtue of Lemma [3, IV.2.20], we can write $S \in \mathcal{M}_{\gamma+3}$. Hence we have

$$S \in (\mathcal{A}_{\gamma+3} \cap \mathcal{M}_{\gamma+3}). \tag{2}$$

Suppose that $SyS^{-1} \notin \mathcal{K}_\beta$ for some rank β. Let β be the minimal rank with this property. By Lemma [3, IV.1.19], there is a normalized occurrence $W \in \text{Norm}(\beta, SyS^{-1}, n - 217)$ in SyS^{-1}. Let $W = P * H * Q$. Since $S \in \mathcal{M}_{\gamma+1}$, we can assume that $\beta \leqslant \gamma$.

According to Lemma [3, IV.1.18], the subwords S and S^{-1} cannot contain more than $(n + 1)/2 + 42$ segments. Hence the elementary word H of rank β is of the form $H = S_1 y S_2$, where y is the central letter of the word SyS^{-1}, S_1 is a suffix of S,

and S_2 is a prefix of S^{-1}. Here each of the two subwords S_1 and S_2 of the word H contains not more than $(n+1)/2+42$ segments and hence not less than

$$n-217-\left(\frac{n+1}{2}+43\right)=\frac{n-1}{2}-260\geqslant 2p$$

segments. These occurrences of the two p-powers S_1 and S_2 are compatible, because they have common continuation W. Then, by virtue of Lemma [3, II.5.17], the subwords S_1 and S_2 must be related. Without loss of generality, we can assume that S_1 is not longer than S_2. In that case, S_1^{-1} must coincide with some suffix of S_2. Hence we see that the elementary p-powers S_1 and S_1^{-1} are related, but this contradicts [3, II.5.22]. Hence our assumption $SyS^{-1}\notin\mathcal{K}_\beta$ was false. Thus, we have proved that

$$SyS^{-1}\in\mathcal{K}_i \qquad \text{for any rank} \quad i. \tag{3}$$

This means that SyS^{-1} is a reduced word in all ranks.

By virtue of Lemma [3, VI.2.8] and of relations $D\in\mathcal{A}_{\gamma+1}$ and (3), using equality (1), we obtain the following equivalence:

$$D\overset{\gamma}{\sim}SyS^{-1}. \tag{4}$$

By virtue of [3, IV.1.7] the single active kernel V of rank γ with period A in the word D contains $>n/3-42$ and $<2n/3$ periods. Then, by virtue of [3, IV.2.2], the image of V in the word SyS^{-1}, $V_1=f_\gamma(V;D,SyS^{-1})$, must also be a single active kernel of rank γ, which by virtue of [3, IV.2.12], must contain not less than $n/3-86>q+4p$ segments.

But by [3, II.5.21] and [3, II.5.2] that kernel must be contained almost entirely in either $*S*yS^{-1}$ or in $Sy*S^{-1}*$. In that case, by [3, III.2.24] and [3, IV.1.5], the word SyS^{-1} itself must have one active kernel of rank γ, whence it follows by [3, IV.2.17] that D has two active kernels of rank γ. This is a contradiction. Hence Case 1) is impossible.

Consider Case 2), when the word $D=aA^t$ conjugates to some power C^l of an elementary period C of some rank.

Suppose that $D=TC^lT^{-1}$. Using the equations $aA^t=D$ and $C^n=1$ in F, we obtain

$$(aA^t)^n=D^n=TC^{ln}T^{-1}=1.$$

Since $A^t \in N$, we see that $a^n \in N$ for an arbitrary nontrivial element a from any group G_i, $(i \in I)$.

Thus, we have proved that the inclusion $G_i^n \subset N$ holds for any $i \in I$.

Now it is sufficient to refer once more to Theorem 7 in [2]. According to that theorem, an arbitrary element g of the group F is conjugate in F either to some element $a \in G_i$ or to some power of some elementary period of some rank.

If the element g conjugates to some element $a \in G_i$, then the required relation $g^n \in N$ follows from the relation $a^n \in N$ for any $a \in G_i$, which was proved above.

If the element g of the group F conjugates to some power of some elementary period of some rank, then the relation $g^n = 1$ holds in F by definition and hence we have $g^n \in N$. Theorem 3 is proved. □

Not residually finite Hopfian groups satisfying a nontrivial identity. A group is said to be *Hopfian* if every surjective endomorphism of the group is an automorphism. According to the classical theorem of Maltsev, all finitely generated residually finite groups are Hopfian. In particular, absolutely free groups and free polynilpotent groups of finite rank are Hopfian. Examples of relatively free solvable not residually finite Hopfian groups have been constructed by Kleiman in [10].

The following is still an open question: Do the free periodic groups $\mathbf{B}(m, n)$ for odd exponents $n \geqslant 665$ satisfy the Hopfian property? Here we give a positive answer to a more general question on the existence of a finitely generated infinite (not simple) Hopfian group satisfying an identity relation of the form $x^n = 1$.

Theorem 4. *If at least one of the factors of the given n-periodic product of groups $G = \prod_{i \in I}^n G_i$ for odd $n \geqslant 665$ does not satisfy the identity relation $x^n = 1$, then G is Hopfian.*

Proof. Suppose that the given n-periodic product of groups G satisfies the assumptions of the theorem. By Theorem 3, the kernel N of an arbitrary noninjective endomorphism α of G contains the subgroup G^n. Then the image $\alpha(G)$ is isomorphic to some quotient group G/G^n of the group G. Hence it satisfies the identity $x^n = 1$. Consequently, the homomorphism α cannot be surjective.

Theorem 4 allows one to construct nonsimple and not residually finite Hopfian groups of bounded period. □

Corrolary 3. *If an odd number $n \geq 665$ is a proper divisor of r, then the n-periodic product of a finite number of cyclic groups of order r is a Hopfian not residually finite and nonsimple group.*

Proof. Let F be n-periodic product of $m > 1$ cyclic groups of order r, where n divides r. Obviously, the free Burnside group $\mathbf{B}(m, n)$ is a homomorphic image of F and the both of these groups are not simple. By Theorem 3, any homomorphism with nontrivial kernel of the group F can be "passed" through $\mathbf{B}(m, n)$.

From the well-known results of Adian [3] and of Zel'manov [11], it follows that the groups $\mathbf{B}(m, n)$ are not residually finite for any odd $n \geq 665$. The Hopfian property of F follows from Theorem 4. $\square$

Acknowledgments. This work was supported in part by the Russian Foundation for Basic Research and the State Committee for Science of the Republic of Armenia (joint Armenian–Russian Research Project no. 13-RF-030).

References

[1] S. I. Adian [Adyan], "Periodic products of groups," in *Trudy Mat. Inst. Steklov*, Vol 142: *Number Theory, Mathematical Analysis, and Their Applications* (Nauka, Moscow, 1976), pp. 3–21 [Proc. Steklov Inst. Math. **142**, 1–19 (1979)].

[2] S. I. Adian [Adyan], "Once More on Periodic Products of Groups and on a Problem of A. I. Maltsev," Mat. Zametki **88** (6), 803–810 (2010) [Math. Notes **88** (6), 771–775 (2010)].

[3] S. I. Adian [Adyan], *The Burnside Problem and Identities in Groups* (Nauka, Moscow 1975; Ergeb. Math. Grenzgeb., vol. 95, Springer-Verlag, Berlin–New York, 1979).

[4] P. S. Novikov and S. I. Adian [Adyan], "On infinite periodic groups. I, II, III," Izv. Akad. Nauk SSSR Ser. Mat. **32**, 212–244, 251–524, 709–731 (1968) [Math. USSR-Izv. **2** (1), (2), (3), 209–236, 241–479, 665–685 (1968)].

[5] S. I. Adian [Adyan], "On the simplicity of periodic products of groups," Dokl. Akad. Nauk SSSR **241**, 745–748 (1978) [Soviet Math. Dokl. **19**, 910–913 (1978)].

[6] B. H. Neumann, "An essay on free products of groups with amalgamations," Philos. Trans. Roy. Soc. London. Ser. A. **246**, 503–554 (1954).

[7] B. H. Neumann and H. Neumann, "Embedding theorems for groups," J. London Math. Soc. **34**, 465–479 (1959).

[8] V. S. Atabekyan, "On normal subgroups in the periodic products of S. I. Adyan," in *Trudy Mat. Inst. Steklov*, Vol. 274: *Algorithmic Aspects of Algebra and Logic*, Collection of papers dedicated to Academician S. I. Adyan on the occasion of his 80th birthday (MAIK Nauka/Interperiodica, Moscow, 2011), pp. 15–31 [Proc. Steklov Inst. Math. **274** (1), 9–24 (2011)].

[9] V. S. Atabekyan, "On CEP-subgroups of n-periodic products," Izv. Nats. Akad. Nauk Armenii Mat. **46** (5), 15–24 (2011) [J. Contemp. Math. Anal. **46** (5), 237–242 (2011)].

[10] Yu. G. Kleiman, "On identities in groups," Trudy Moskov. Mat. Obshch. **44**, 62–108 (1982).

[11] E. I. Zelmanov, "Solution of the restricted Burnside problem for groups of odd exponent," Izv. Akad. Nauk SSSR Ser. Mat. **54** (1), 42–59 (1990) [Math. USSR-Izv. **36** (1), 41–60 (1991)].